

HIGHLIGHTS OF SYMMETRY GROUPS

Giampiero Esposito and Giuseppe Marmo

INFN, Sezione di Napoli, and Dipartimento di Scienze Fisiche, Complesso Universitario di Monte S. Angelo, Via Cintia, Edificio N', 80126 Napoli, Italy

Abstract. The concepts of symmetry and symmetry groups are at the heart of several developments in modern theoretical and mathematical physics. The present paper is devoted to a number of selected topics within this framework: Euclidean and rotation groups; the properties of fullerenes in physical chemistry; Galilei, Lorentz and Poincaré groups; conformal transformations and the Laplace equation; quantum groups and Sklyanin algebras. For example, graphite can be vaporized by laser irradiation, producing a remarkably stable cluster consisting of 60 carbon atoms. The corresponding theoretical model considers a truncated icosahedron, i.e. a polyhedron with 60 vertices and 32 faces, 12 of which are pentagonal and 20 hexagonal. The C_{60} molecule obtained when a carbon atom is placed at each vertex of this structure has all valences satisfied by two single bonds and one double bond. In other words, a structure in which a pentagon is completely surrounded by hexagons is stable. Thus, a “cage” in which all 12 pentagons are completely surrounded by hexagons has optimum stability. On a more formal side, the exactly solvable models of quantum and statistical physics can be studied with the help of the quantum inverse problem method. The problem of enumerating the discrete quantum systems which can be solved by the quantum inverse problem method reduces to the problem of enumerating the operator-valued functions that satisfy an equation involving a fixed solution of the quantum Yang–Baxter equation. Two basic equations exist which provide a systematic procedure for obtaining completely integrable lattice approximations to various continuous completely integrable systems. This analysis leads in turn to the discovery of Sklyanin algebras.

1.1 Introduction to symmetry groups

Symmetry pervades all our descriptions, modelling or understanding of natural phenomena. From microscopic physics to cosmology, from chemistry to biology, symmetry properties or considerations are ubiquitous in scientific research.

In ancient times, symmetry meant mostly “harmony in the proportions”, and in modern natural science it emerges mostly as the “invariance with respect to a transformation group”. In abstract terms, any set \mathcal{S} is associated with a group $\text{Aut } \mathcal{S}$, the family of all invertible maps $\varphi : \mathcal{S} \rightarrow \mathcal{S}$, which may be composed:

$$(\varphi_1 \cdot \varphi_2)(s) = \varphi_1(\varphi_2(s)), \quad (1.1.1)$$

and the composition is associative, i.e.

$$(\varphi_1 \cdot \varphi_2) \cdot \varphi_3 = \varphi_1 \cdot (\varphi_2 \cdot \varphi_3), \quad (1.1.2)$$

it allows for the identity transformation

$$\mathbb{I} : s \rightarrow s, \quad \forall s \in \mathcal{S}, \quad (1.1.3)$$

and an inverse φ^{-1} such that

$$\varphi(\varphi^{-1}(s)) = \varphi^{-1}(\varphi(s)) = \mathbb{I} \cdot s = s. \quad (1.1.4)$$

Any subset, pattern, configuration \mathcal{P} in \mathcal{S} defines a subgroup of $\text{Aut } \mathcal{S}$; by selecting only those transformations which preserve or keep invariant \mathcal{P} , for any $p \in \mathcal{P} \subset \mathcal{S}$ we consider only those transformations $\varphi \in \text{Aut } \mathcal{S}$ such that $\varphi(p) \in \mathcal{P}$, $\forall p \in \mathcal{P}$. This subgroup is never empty because it contains at least the identity transformation which constitutes the smallest subgroup in $\text{Aut } \mathcal{S}$. In addition to subsets, one may select subgroups of $\text{Aut } \mathcal{S}$, by requiring that properties or relations among pairs or more points are preserved. This strict connection between configurations and transformations was explicitly introduced by F. Klein and S. Lie. In the Erlangen program (1872) of F. Klein, a geometric theory is defined as the study of those properties of the space, \mathcal{S} , and of its subsets (figures) which are preserved with respect to a selected subgroup of $\text{Aut } \mathcal{S}$. Similarly, in physics, starting with Galileo and culminating with Einstein, every physical theory carries with it its “covariance” group.

It was P. Curie who (translated and) incorporated into physics the role of symmetry as a working tool in our formalization of the external world, by stating that “symmetries

in the cause should be reflected in the effects, and the lack of symmetries in the effects should be searched for in the causes". Of course, the occurrence of "natural symmetry breaking" would require a better formulation of Curie's principle. Symmetries as invariance with respect to a selected group of transformations are dealt with in physics as invariance principles. Explicitly we find the requirement of invariance in special relativity with respect to the Poincaré group (see below), this brings in the notion of reference system, their equivalence or lack of.

The use of transformations for classification purposes has introduced in physics the transformation method, i.e. a given system is transformed into an equivalent one considered as a model system or a "normal form" of it. In heuristic terms one may say that to similar problems there correspond similar solutions (similar means *connected by symmetries*). In analytic mechanics, this approach has brought in the use of canonical transformations, Hamilton and Jacobi. In this approach, evolution itself, the dynamics, is presented as a 1-parameter group of transformations. In modern physics, the formulation of special relativity and general relativity on the one hand, and the extension of canonical transformations in the quantum setting due to P. Dirac on the other hand, has intimately connected physics, geometry and transformation groups. In mathematics, this research line has culminated in the study of Lie groups, Lie algebras, representation theory and, in particular, unitary representations.

By definition, a *symmetry* of a dynamical system is any transformation which maps bijectively the set of solutions onto itself. Although not strictly needed, one can add the further requirement that symmetries should preserve the parametrization of solutions. In the case of a dynamical system obeying Newton type equations of motion, i.e.

$$\frac{dq^i}{dt} = u^i, \quad \frac{du^i}{dt} = f^i,$$

one may or may not require the relation between positions and velocities to be preserved. In the former case, symmetry transformations are further qualified as *point symmetries*.

According to the Noether Theorem, if the Lagrangian of a physical system is invariant under a 1-parameter group of transformations φ_τ on the tangent bundle of configuration space, then there exists a constant of motion which can immediately be associated with this invariance, or rather with the *symmetry group* φ_τ . It should be remarked that this association is Lagrangian dependent, i.e. if the equations of motion admit alternative Lagrangian descriptions, the conservation of angular momentum, for instance, may give rise to different invariance groups.

1.2 Simple examples: Euclidean and rotation groups

A first important example of symmetry group is here provided by the Euclidean group on \mathbf{R}^3 , i.e. the group of affine transformations which preserve the length of vectors (the length of vectors being evaluated with an Euclidean metric g). It contains translations and linear homogeneous transformations which satisfy the condition

$$g(Tx, Ty) = g(x, y). \quad (1.2.1)$$

By using a basis for \mathbf{R}^3 , e.g. orthonormal vectors represented by row or column vectors of the form $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, a generic matrix

$$R \equiv \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix} \quad (1.2.2)$$

represents a rotation if

$$g(\vec{\alpha}, \vec{\alpha}) = g(\vec{\beta}, \vec{\beta}) = g(\vec{\gamma}, \vec{\gamma}) = 1, \quad (1.2.3)$$

$$g(\vec{\alpha}, \vec{\beta}) = g(\vec{\alpha}, \vec{\gamma}) = g(\vec{\beta}, \vec{\gamma}) = 0. \quad (1.2.4)$$

It is easy to show that these 6 conditions imply that R is a rotation matrix if and only if its transpose equals its inverse. Now we see that

$$(R_1 R_2)^t (R_1 R_2) = R_1 R_2^t R_2^t R_1 = \mathbb{I}.$$

Thus, the product of two rotations is again a rotation, and the identity matrix is of course a rotation. By virtue of all these properties, the set of rotations is a subgroup of the group of linear invertible matrices. The rotation group is denoted by $O(3)$. It is three-dimensional because one has to subtract from the dimension of \mathbf{R}^9 the number of relations in Eqs. (1.2.3) and (1.2.4). Moreover, $\det R = \pm 1$, and because \det is a continuous function, the matrices with determinant 1 and those with determinant -1 are two different connected components. The set of rotations has only two components: those expressed by matrices having -1 determinant, which do not form a subgroup since the identity does not belong to them, and those expressed by matrices having $+1$ determinant. The latter form the group $SO(3)$ and have the important property of preserving volumes.

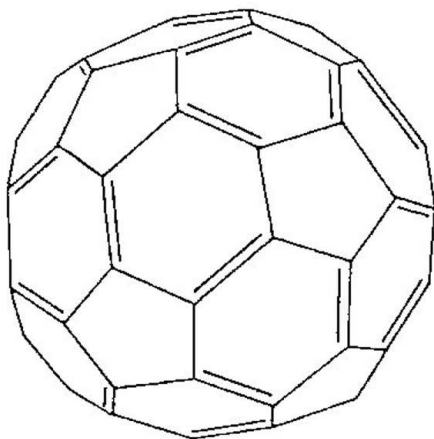
References

- [1] Curie, P. 1894. Sur la symmetrie dans les phénomènes physiques. Symmetrie d'un champ électrique et d'un champ magnétique. *Journal de Physique*, 3
- [2] Dirac, P.A.M. 1958. *The Principles of Quantum Mechanics*, Oxford: Clarendon Press
- [3] Marmo, G., Saletan, E.J., Simoni, A. & Vitale, B. 1983. *Dynamical Systems. A Differential Geometric Approach to Symmetry and Reduction*, New York: Wiley
- [4] Weyl, H. 1952. *Symmetry*, Princeton: Princeton University Press
- [5] Wigner, E.P. 1959. *Group Theory and Its Application to the Quantum Mechanics of Atomic Spectra*, New York: Academic

2. Applications to physical chemistry

A very relevant modern application of symmetry concepts in physics is given by fullerenes. Until 1985, the chemical element Carbon was only known to exist in two forms, i.e. diamond and graphite. This changed when Kroto and co-workers discovered an entirely new form of carbon, which became known as C_{60} or the fullerene molecule. The original discovery of C_{60} was produced from the laser ablation of graphite. Since then, other methods of production have been developed. It is also thought that isolated C_{60} molecules may be found in stars and interstellar media.

The fullerene molecule (see Figure) consists of 60 carbon atoms arranged in pentagons and hexagons, very like in a standard football. It is also known as Buckminster Fullerene, by virtue of the resemblance of this shape to the geodesic domes designed and built by the architect R. Buckminster Fuller. More precisely, during experiments aimed at understanding the mechanism by which long-chain carbon molecules are formed in interstellar space, graphite was vaporized by laser irradiation, producing a remarkably stable cluster consisting of 60 carbon atoms. To understand what kind of 60-carbon atom structure might give rise to a superstable species, Kroto and co-authors suggested a truncated icosahedron, a polygon with 60 vertices and 32 faces, 12 of which are pentagonal and 20 hexagonal. The C_{60} molecule which results when a carbon atom is placed at each vertex of this structure has all valences satisfied by two single bonds and one double bond, has many resonance structures, and appears to be aromatic. In their investigation, Kroto and co-workers pointed out that, if one considers a tetrahedral diamond structure, the whole surface of the cluster would be covered with unsatisfied valences. This led them to look for another plausible structure which would satisfy all sp^2 valences. Only a spheroidal structure appeared likely to satisfy this criterion, and hence they consulted the studies of Buckminster Fuller.



The Figure shows the proposed structure of C_{60} Buckminsterfullerene, the archetype of the fullerene family. It has t -icosahedral symmetry, as does the modern European football. One of the properties exploited in arriving at this model is the fact that a structure in which a pentagon is completely surrounded by hexagons is stable (Barth and Lawton 1971). Thus, a cage in which all 12 pentagons are completely surrounded by hexagons has optimum stability.

References

- [1] Barth, W.E. & Lawton, R.G. 1971. The synthesis of corannulene. *J. Am. Chem. Soc.*, 93: 1730-45
- [2] Kroto, H.W., Heath, J.R., O'Brien, S.C., Curl, R.F. & Smalley, R.E. 1985. C_{60} : Buckminsterfullerene. *Nature*, 318: 162-63
- [3] Kroto, H.W. 1987. The stability of the fullerenes C_n , with $n = 24, 28, 32, 36, 50, 60$ and 70. *Nature*, 329: 529-31
- [4] Kroto, H.W. 1997. Symmetry, space, stars and C_{60} . *Rev. Mod. Phys.*, 69: 703-22
- [5] Smalley, R.E. 1997. Discovering the fullerenes. *Rev. Mod. Phys.*, 69: 723-30

3. More general groups and related topics

Now we move on to a description of more general groups, which find a wide range of applications in modern theoretical physics.

3.1 Galilei group

The Galilei group expresses the geometric invariance properties of the equations of motion of a non-relativistic classical dynamical system when the system is isolated from external influences. The general Galilei transformation $G(R, \vec{v}, \vec{\xi}, \tau)$ takes a point of space-time with coordinates x_1, x_2, x_3, t to another point with coordinates x'_1, x'_2, x'_3, t' given by

$$\vec{x}' = R\vec{x} + \vec{v}t + \vec{\xi}, \quad (3.1.1)$$

$$t' = t + \tau, \quad (3.1.2)$$

where $R \in SO(3)$, $\vec{\xi}$ and \vec{v} are fixed vectors in \mathbf{R}^3 , and τ is a real constant. The resulting group multiplication law is

$$\begin{aligned} G(R_2, \vec{v}_2, \vec{\xi}_2, \tau_2)G(R_1, \vec{v}_1, \vec{\xi}_1, \tau_1) \\ = G(R_2R_1, R_2\vec{v}_1 + \vec{v}_2, R_2\vec{\xi}_1 + \vec{\xi}_2 + \vec{v}_2\tau_1, \tau_2 + \tau_1). \end{aligned} \quad (3.1.3)$$

The 1-parameter subgroups are rotations about a fixed axis, transformations to frames moving in a fixed direction, displacements of the origin in a fixed direction, and time displacements. These correspond to fixed axes of rotation for R (3 subgroups), fixed directions of \vec{v} (3 subgroups), fixed directions of $\vec{\xi}$ (3 subgroups) and time displacements (1 subgroup). The Galilei group is hence 10-dimensional. The commutation relations for its Lie algebra are

$$\{M_\alpha, M_\beta\} = \varepsilon_{\alpha\beta\gamma}M_\gamma, \quad (3.1.4)$$

$$\{M_\alpha, P_\beta\} = \varepsilon_{\alpha\beta\gamma}P_\gamma, \quad (3.1.5)$$

$$\{M_\alpha, G_\beta\} = \varepsilon_{\alpha\beta\gamma}G_\gamma, \quad (3.1.6)$$

$$\{H, G_\alpha\} = -P_\alpha, \quad (3.1.7)$$

$$\{M_\alpha, H\} = \{P_\alpha, G_\beta\} = \{G_\alpha, G_\beta\} = \{P_\alpha, P_\beta\} = \{P_\alpha, H\} = 0, \quad (3.1.8)$$

where the M_α is the infinitesimal generator of rotations about the α axis, G_α of Galilei transformations to frames moving in the fixed α -direction, P_α of displacements of the origin in the α -direction, and H of time displacements.

3.2 Lorentz and Poincaré groups

The Lorentz group is defined with the Euclidean metric g being replaced by the Minkowski metric η in Eq. (1.2.1). One still has $\eta(Tx, Ty) = \eta(x, y)$, and by virtue of the signature of the metric, if one writes

$$T \equiv \begin{pmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_0 & \beta_1 & \beta_2 & \beta_3 \\ \gamma_0 & \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_0 & \delta_1 & \delta_2 & \delta_3 \end{pmatrix} \quad (3.2.1)$$

one finds

$$\eta(\alpha, \alpha) = -1, \quad \eta(\beta, \beta) = \eta(\gamma, \gamma) = \eta(\delta, \delta) = 1, \quad (3.2.2)$$

$$\eta(\rho, \sigma) = 0 \quad \forall \rho \neq \sigma, \quad (3.2.3)$$

for all $\rho, \sigma = \alpha, \beta, \gamma, \delta$. These 10 independent conditions determine a 6-parameter group in the 16-dimensional space of 4×4 matrices. On adding to this the linear non-homogeneous transformations known as space-time translations one gets the Poincaré group. More precisely, this is the abstract group isomorphic to the geometric group of transformations of a ‘world point’, and it can be defined independently of any split of the space-time manifold.

In the applications, one is interested in the following realization of the group in terms of transformations of a world point:

(i) Space-time displacements: $\vec{r}' = \vec{r} + \vec{a}$, $t' = t$ or $\vec{r}' = \vec{r}$, $t' = t + b$.

(ii) Moving frames:

$$\vec{r}' = \frac{(\vec{v} \wedge \vec{r}) \wedge \vec{v}}{v^2} + \frac{\vec{v}}{v^2} \frac{\vec{v} \cdot \vec{r} - v^2 t}{\sqrt{1 - v^2}}, \quad |\vec{v}| < 1, \quad (3.2.4)$$

$$t' = \frac{t - \vec{v} \cdot \vec{r}}{\sqrt{1 - v^2}}. \quad (3.2.5)$$

(iii) Space rotations: $\vec{r}' = R \vec{r}$, $t' = t$.

If we represent the general transformation by $T(\vec{a}, b, \vec{v}, R)$, with the convention

$$T(\vec{a}, b, \vec{v}, R) = T(\vec{a}, 0, \vec{0}, \mathbb{I})T(\vec{0}, b, \vec{0}, \mathbb{I})T(\vec{0}, 0, \vec{v}, \mathbb{I})T(\vec{0}, 0, \vec{0}, R), \quad (3.2.6)$$

the effect of a general transformation on (\vec{r}, t) is

$$\vec{r}' = \vec{a} + \frac{(\vec{v} \wedge R\vec{r}) \wedge \vec{v}}{v^2} + \frac{\vec{v}}{v^2} \frac{\vec{v} \cdot R\vec{r} - v^2 t}{\sqrt{1 - v^2}}, \quad (3.2.7)$$

$$t' = b + \frac{t - \vec{v} \cdot R\vec{r}}{\sqrt{1 - v^2}}. \quad (3.2.8)$$

Actually the group that we have so far defined is the connected Lie sub-group G_c , since we have not considered the inversion operations

$$R_p : \vec{r}' = -\vec{r}, \quad t' = t,$$

$$R_T : \vec{r}' = \vec{r}, \quad t' = -t,$$

$$R_T R_p = R_S : \vec{r}' = -\vec{r}, \quad t' = -t.$$

The full group quotiented by the invariant sub-group G_c , connected component containing the identity, yields a factor group described by \mathbb{I}, P, T, TP . To implement this quotient group as a sub-group of the original one, it is mandatory to introduce, at this stage, a split of the space-time manifold (see Eqs. (3.2.4)–(3.2.8)), and its realization depends on the split into space and time.

One can exhibit an inverse for every element $T(\vec{a}, b, \vec{v}, R)$ and show that a multiplication exists which is associative so that one has a group; moreover the composition functions $z_r \equiv f_r(x, y)$, with $x = (\vec{a}, b, \vec{v}, R)$ etc. are differentiable functions of x and y , so that the system so defined is a Lie group with 10 parameters, which is clearly non-commutative. It is customary to exhibit the 10 generators in the Lie algebra as functions with the help of a Poisson bracket on $T^*\mathbf{R}^4$, the cotangent bundle of \mathbf{R}^4 , as:

$$\vec{p} = \{p_j\}, \quad j = 1, 2, 3; \text{ space displacement,}$$

$$H = P_0, \text{ time displacement,}$$

$$\vec{K} = \{K_j\}, \quad j = 1, 2, 3; \text{ moving frames,}$$

$$\vec{J} = \{J_j\}, \quad j = 1, 2, 3; \text{ space rotations.}$$

The composition law for the Poincaré group is then equivalent to the following bracket relations for the generators:

$$\{P_j, P_k\} = 0, \quad \{P_j, H\} = 0, \quad \{K_j, P_k\} = \delta_{jk}H, \quad \{J_j, P_k\} = \varepsilon_{jkl}P_l, \quad (3.2.9)$$

$$\{K_j, H\} = P_j, \quad \{J_j, H\} = 0, \quad (3.2.10)$$

$$\{K_j, K_k\} = -\varepsilon_{jkl}J_l, \quad \{J_j, K_k\} = \varepsilon_{jkl}K_l, \quad (3.2.11)$$

$$\{J_j, J_k\} = \varepsilon_{jkl}J_l. \quad (3.2.12)$$

Experience with the rotation, Euclidean and Galilei groups tells us that, as applied to dynamical systems, generally the infinitesimal generators (i.e. the elements of the Lie algebra) have an immediate physical interpretation. We then expect a similar situation to arise for the Poincaré group also. By the close correspondence with the Galilei group, we identify \vec{P} with linear momentum, H with the energy and \vec{J} with the angular momentum. We may also consider \vec{K} to be a relativistic ‘moment’. Bearing this in mind we may now look at the Poisson-bracket relations; the generators now play a dual role. On the one hand they represent generators of infinitesimal transformations; on the other hand they represent physical quantities. Thus, the Poisson-bracket relations

$$\{J_j, P_k\} = \varepsilon_{jkl} P_l$$

may be interpreted either as stating that the linear momentum transforms like a vector under a rotation (i.e. the increment in \vec{P} in an infinitesimal rotation is at right angles to itself and the axis of rotation); or as stating that the angular momentum increases by a quantity proportional to the normal component of momentum under a displacement. Similarly, the first of the relations (3.2.10) may be taken to mean that the energy changes on transforming to a moving frame by a quantity proportional to the component of linear momentum along the direction of relative velocity. Equally well it may be taken to mean that the relativistic moment \vec{K} is not a constant but changes linearly with respect to time by a quantity proportional to the linear momentum.

Poincaré group and gauge transformations

In the investigation of Maxwell’s electrodynamics on \mathbf{R}^3 , we are familiar with the decomposition of the vector potential A into its longitudinal part A^L and transverse part A^T (this is a corollary of the Kodaira decomposition theorem of the space of differential forms, bearing in mind that no harmonic forms exist on a contractible manifold), such that $dA^L = 0$ and $\delta A^T = 0$, d and δ denoting exterior differentiation and co-differentiation, respectively. One may then impose a supplementary condition, more frequently called gauge choice, e.g. Lorenz or Coulomb or axial or temporal, to achieve a particular form of the field equations expressed in terms of the potential (and also to ensure that the operator occurring in the classical theory of small disturbances is invertible). At that stage, all calculations rely on a gauge choice made in a given reference frame. However, to obtain a Poincaré-covariant formulation one needs a transformation that takes us from

$$\delta A^T = 0 \tag{3.2.13}$$

in a given frame, to the same form

$$\delta' A'^T = 0 \quad (3.2.14)$$

in a different frame. For this to be the case, it is necessary *to change the action of the Poincaré group on the vector potential*. Of course, we can transform the vector potential by adding to it a closed 1-form so that the resulting electromagnetic field is not affected.

If X is any vector field in the Poincaré algebra, one can associate with it an operator D_X the action of which is given by

$$D_X A = L_X A + df_X. \quad (3.2.15)$$

It can be proved that the appropriate functions can be so chosen that D_X provides a representation of the Poincaré algebra

$$D_X D_Y - D_Y D_X = D_{[X,Y]}. \quad (3.2.16)$$

This is called a *gauge-dependent representation of the Poincaré algebra*, which maps Eq. (3.2.13) into Eq. (3.2.14).

References

- [1] Sudarshan, E.C.G. & Mukunda, N. 1974. *Classical Dynamics: A Modern Perspective*, New York: Wiley.
- [2] DeWitt, B.S. 2003. *The Global Approach to Quantum Field Theory*, Oxford: Oxford University Press.
- [3] Barut, A.O. 1980. *Electrodynamics and Classical Theory of Fields and Particles*, New York: Dover.
- [4] Marmo, G. 1986. *Differential Forms and Electrodynamics* (Lectures at Shanxi University).

3.3 Conformal Transformations

Given the space-time manifold (M, g) , a *conformal rescaling* of the metric $g = g_{ab}dx^a \otimes dx^b$ is a map

$$g_{ab} \rightarrow \hat{g}_{ab} = \Omega^2 g_{ab}, \quad (3.3.1)$$

where Ω is a smooth and positive scalar function on M . If g_{ab} is the Minkowski metric $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$, the new metric $\hat{\eta}_{ab}$ is not flat in general, but there are exceptions: the Riemann tensor obtained from $\hat{\eta}_{ab}$ when Ω is constant or equal to $(x^a x_a)^{-1}$ does indeed vanish.

If (M_1, g) and (M_2, h) are two pseudo-Riemannian manifolds, then a diffeomorphism

$$f : (M_1, g) \rightarrow (M_2, h)$$

is said to be a *conformal mapping* if the metric induced on M_2 by f is a conformal rescaling of the given metric on M_2 . Two standard but important examples are as follows.

(i) If both M_1 and M_2 are Minkowski space-time, and f is the map

$$f : x^a \rightarrow kx^a, \quad (3.3.2)$$

with k a positive constant, then the induced metric $\hat{\eta}_{ab}$ is related to the original metric by $\hat{\eta}_{ab} = k^{-2}\eta_{ab}$. Such a conformal mapping is called a *dilation*.

(ii) Let M_1 be Minkowski space-time minus the null cone of p^a , while M_2 is chosen to be Minkowski space-time minus the null cone of the origin. Define the map f by

$$f : x^a \rightarrow \frac{p^a - x^a}{(p_b - x_b)(p^b - x^b)}. \quad (3.3.3)$$

Such a map is a diffeomorphism, and the induced metric $\hat{\eta}_{ab}$ on M_2 is related to the original metric η_{ab} on M_1 by $\hat{\eta}_{ab} = \Omega^2 \eta_{ab}$, with $\Omega(x) \equiv (x^a x_a)^{-1}$. The resulting conformal mapping is called an *inversion*. The next step consists in adjoining to Minkowski space-time a null cone at infinity, here denoted by \mathcal{I} . Hence one gets a compact manifold M containing Minkowski space-time, and the inversion map f can be extended to a conformal map from M to itself.

The Lie group of conformal mappings from M to itself is the *conformal group* and is denoted by $C(1, 3)$. It is 15-dimensional in 4 space-time dimensions, being generated by the Poincaré transformations (6 parameters from Lorentz and 4 from translations), the dilations (1 parameter) and the inversions (4 parameters).

When conformal transformations are considered, two meanings can be attached to the statement that a field theory is conformally invariant. A first option is to require that it should be possible to assign a conformal weight to each field in such a way that the resulting field equations are invariant under arbitrary conformal rescalings of the metric. The fields of the theory (either scalar, or tensor or spinor) are said to have conformal weight k if they transform according to $\varphi \rightarrow \Omega^k \varphi$ under a conformal rescaling of the metric. The consideration of Minkowski space-time is sufficient to show an important property: since under a dilation the d'Alembert operator \square transforms as $\square \rightarrow k^{-2} \square$, the field equation $(\square + m^2)\varphi = 0$ is not invariant unless the mass parameter vanishes. Thus, to be conformally invariant, a field theory has to involve only massless fields.

To define conformal invariance one can however require that the field theory should be invariant under the conformal group $C(1,3)$. This is the second possible definition. If a theory is both Poincaré-invariant and invariant under conformal rescalings, it is then conformally invariant in this second sense as well. This holds because the Poincaré transformations become conformal transformations according to any other conformally rescaled flat metric, and the resulting conformal transformations, jointly with Poincaré transformations, generate the whole conformal group.

References

- [1] Penrose, R. 1974. Relativistic symmetry groups, in *Group Theory in Nonlinear Problems*, ed. A.O. Barut, Dordrecht: Reidel Publishing Company
- [2] Ward, R.S. & Wells, R.O. 1990. *Twistor Geometry and Field Theory*, Cambridge: Cambridge University Press

3.4 Laplace Equation

Let us consider functions u depending on n variables x_1, x_2, \dots, x_n in a domain G of \mathbf{R}^n with boundary Γ . The differential equation

$$\Delta u = \sum_{l=1}^n \frac{\partial^2 u}{\partial x_l^2} = 0 \quad (3.4.1)$$

is called the Laplace equation, and its solutions are called *harmonic functions*. The harmonic functions belong therefore to the kernel of the Laplace operator Δ .

Solutions of the Laplace equation which have continuous second derivatives in an open, connected and bounded region G of space are called *regular* in G . In 2 dimensions the general solution of the Laplace equation is the real part of any analytic function of the complex variable $x + iy$. In 3 dimensions it remains easy to construct solutions which depend on arbitrary functions. For example, let $f(w, t)$ be analytic in the complex variable w for fixed real t . When the variable t is allowed to take arbitrary values, both the real and imaginary parts of the function

$$u \equiv f(z + ix \cos t + iy \sin t, t)$$

of the real variables x, y, z are solutions of the Laplace equation. Further solutions may be obtained by superposition via an integral representation, i.e.

$$u = \int_a^b f(z + ix \cos t + iy \sin t, t) dt. \quad (3.4.2)$$

For example, on setting $f(w, t) = w^n e^{iht}$ for some integers n and h , integration from $-\pi$ to $+\pi$ yields the homogeneous polynomials

$$u = \int_{-\pi}^{\pi} (z + ix \cos t + iy \sin t)^n e^{iht} dt = r^n e^{ih\phi} P_{n,h}(\cos \theta), \quad (3.4.3)$$

having introduced polar coordinates in \mathbf{R}^3 , with $P_{n,h}$ the standard notation for Legendre functions.

By transforming to polar coordinates in the plane or in space, the action of the Laplacian becomes

$$\Delta u = \frac{1}{r} \left[\frac{\partial}{\partial r} (r u_r) + \frac{\partial}{\partial \phi} \left(\frac{u_\phi}{r} \right) \right], \quad (3.4.4)$$

$$\Delta u = \frac{1}{r^2 \sin \theta} \left[\frac{\partial}{\partial r} (r^2 u_r \sin \theta) + \frac{\partial}{\partial \theta} (u_\theta \sin \theta) + \frac{\partial}{\partial \phi} \left(\frac{u_\phi}{\sin \theta} \right) \right], \quad (3.4.5)$$

respectively. These formulae, and their extension to \mathbf{R}^n , make it possible to prove the following theorem: *if $u(x_1, \dots, x_n)$ is a regular harmonic function in the domain G , the function*

$$v(x_1, \dots, x_n) = r^{-(n-2)} u \left(\frac{x_1}{r^2}, \dots, \frac{x_n}{r^2} \right) \quad (3.4.6)$$

also satisfies the Laplace equation and is regular in the region G' obtained from G by inversion with respect to the unit sphere. We therefore learn that, apart from the factor r^{2-n} , the harmonic character of a function is invariant under inversions with respect to spheres. Moreover, the harmonic property is retained completely under similarity transformations, translations, and simple reflections across planes.

Let the function u be regular and harmonic in a bounded domain G . If we invert G with respect to a sphere of unit radius whose centre lies in G , the interior of G is carried into the exterior G' of the inverted boundary surface Γ' . The harmonic function in Eq. (3.4.6) is then called regular in this exterior region G' . Regularity in a domain G extending to infinity is therefore defined as follows: the domain G is inverted with respect to a sphere with centre outside of G , so that G is transformed into a bounded domain G' . By definition, the harmonic function u is called regular in G if the above function v is regular in G' . In particular, u is said to be *regular at infinity* if G contains a neighbourhood of the point at infinity and a value is assigned to the function u at the point at infinity such that v is regular in the bounded domain G' . These definitions imply that the constant function is regular at infinity in the plane, but not in spaces of 3 or more dimensions. In space, for arbitrary values of a , the functions

$$u \equiv 1 - a + \frac{a}{r}$$

are harmonic outside the unit sphere and take the boundary value $u = 1$ on the sphere. But $u = \frac{1}{r}$ is the only function of this family which is regular in the region exterior to the unit sphere.

For any number n of dimensions, the only solutions of the Laplace equation which depend only on the distance r of the point x from a fixed point ξ , are (up to arbitrary multiplicative and additive constants) the functions (ω_n being the surface area of the unit sphere S^n)

$$\gamma(r) = \frac{1}{(n-2)\omega_n} r^{2-n}, \quad \text{if } n > 2, \quad (3.4.7)$$

$$\gamma(r) = \frac{1}{2\pi} \log \frac{1}{r}, \quad \text{if } n = 2. \quad (3.4.8)$$

These exhibit the so-called characteristic singularity at $r = 0$. Every solution of the Laplace equation in the domain G of the form (here $r^2 \equiv \sum_{l=1}^n (x_l - \xi_l)^2$)

$$\psi(x_1, \dots, x_n; \xi_1, \dots, \xi_n) = \gamma(r) + w, \quad (3.4.9)$$

for ξ inside G and w regular, is said to be a *fundamental solution* with a singularity at the parameter point ξ .

Rotational invariance of the Laplace equation (3.4.1) is proved in an elegant way by pointing out that the symbol σ (or characteristic polynomial) of the Laplace operator therein reads

$$\sigma(k) = k_{x_1}^2 + \dots + k_{x_n}^2, \quad (3.4.10)$$

which is the squared length of the n -component vector \vec{k} in momentum space, and it is well known that rotations preserve the length of vectors. Note that, when writing Eq. (3.4.10), we end up defining the Laplace operator with a minus sign in front of all second derivatives. This has two related advantages: (i) it yields a Laplace operator with a spectrum bounded from below on compact Riemannian manifolds; (ii) it leads to a symbol which is a positive-definite quadratic form.

Operators which differ from the Laplacian by the addition of a potential term are said to be of Laplace type. In modern field theories of fundamental interactions, these operators act on sections of vector bundles V over a base space which is a Riemannian manifold (M, g) , and read as

$$P \equiv -g^{ab} \nabla_a \nabla_b - E, \quad (3.4.11)$$

where ∇ is the connection on V , and E is an endomorphism of V (i.e. the potential term in the operator of Laplace type). Equation (3.4.11) expresses the general form of the operator acting on the gauge field whenever linear covariant gauges are used for (path-integral) quantization.

References

- [1] Kellogg, O.D. 1954. *Foundations of Potential Theory*, New York: Dover
- [2] Courant, R. & Hilbert, D. 1961. *Methods of Mathematical Physics. II. Partial Differential Equations*, New York: Interscience
- [3] Garabedian, P.R. 1964. *Partial Differential Equations*, New York: Chelsea

- [4] Avramidi, I.G. & Esposito, G. 1999. Gauge theories on manifolds with boundary. *Communications in Mathematical Physics*, 200: 495-543

3.5 Quantum Groups

Several current developments in group and field theory rely on the concept of quantum group. Such a name is an abuse of language, since one actually deals with a Hopf algebra consisting of the following structures:

- (i) A unital algebra H over the complex field \mathcal{C} (i.e. a linear space H endowed with multiplication, say $m : H \times H \rightarrow \mathcal{C}$, and unity maps which are complex-linear, satisfy the associativity of multiplication and the existence of unit element);
- (ii) A coproduct $\Delta : H \rightarrow H \otimes H$ and counit $\varepsilon : H \rightarrow \mathcal{C}$ forming a coalgebra, with Δ, ε algebra homomorphisms;
- (iii) An antipode $S : H \rightarrow H$ such that $(S \otimes \text{id})\Delta = i\varepsilon = (\text{id} \otimes S)\Delta$.

A coalgebra is just like an algebra but with the arrows on the maps occurring in the axioms reversed. Thus, the coassociativity and counity axioms are

$$(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta, \quad (3.5.1)$$

$$(\varepsilon \otimes \text{id})\Delta = (\text{id} \otimes \varepsilon)\Delta = \text{id}. \quad (3.5.2)$$

The antipode plays a role that generalizes the concept of group inversion. Other than that the only new mathematical structure that the reader has to contend with is the coproduct Δ and its associated counit. There are several ways of interpreting the meaning of this, depending on our point of view. If the quantum group is like the enveloping algebra $U(g)$ generated by a Lie algebra g , one should think of Δ as providing the rule by which actions extend to tensor products. Thus, $U(g)$ is trivially a Hopf algebra with

$$\Delta\xi = \xi \otimes 1 + 1 \otimes \xi, \quad \forall \xi \in g, \quad (3.5.3)$$

which says that when a Lie algebra element ξ acts on tensor products it does so by ξ in the first factor and then ξ in the second factor. Similarly it says that when a Lie algebra acts on an algebra it does so as a derivation. On the other hand, if the quantum group is like a coordinate algebra $\mathcal{C}[G]$, then Δ expresses the group multiplication and ε the group identity element e . Thus, if $f \in \mathcal{C}[G]$, the coalgebra is

$$(\Delta f)(g, h) = f(gh), \quad \forall g, h \in G, \quad \varepsilon f = f(e), \quad (3.5.4)$$

at least for suitable choices of f . In other words, it expresses the group product $G \times G \rightarrow G$ by a map in the reversed direction in terms of coordinate algebras.

From yet another point of view Δ simply makes the dual H^* also into an algebra. Hence a Hopf algebra is basically an algebra such that the dual H^* is also an algebra, in a compatible way. For every finite-dimensional H there is a dual H^* , and similarly in the infinite-dimensional case, where one has to distinguish between algebraic and topological duals. Among the many examples that can be given, we here mention only two, for length reasons.

(i) The Planck scale quantum group generated by position x and momentum p with commutation relations (here l is a parameter having dimension length)

$$[x, p] = i\hbar(1 - e^{-x/l}), \quad (3.5.5)$$

and coproduct

$$\Delta x = x \otimes 1 + 1 \otimes x, \quad (3.5.6)$$

$$\Delta p = p \otimes e^{-x/l} + 1 \otimes p. \quad (3.5.7)$$

(ii) Quantum groups may be viewed as deformed enveloping algebras. The simplest example is the quantum group $U_q(su_2)$ with generators H, X_{\pm} and defining relations

$$[H, X_{\pm}] = \pm 2X_{\pm}, \quad (3.5.8)$$

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}, \quad (3.5.9)$$

and coproduct

$$\Delta X_{\pm} = X_{\pm} \otimes q^{H/2} + q^{-H/2} \otimes X_{\pm}, \quad (3.5.10)$$

$$\Delta H = H \otimes 1 + 1 \otimes H. \quad (3.5.11)$$

The coproduct Δ here is a deformation of the usual additional one, which is recovered as $q \rightarrow 1$. The deformation modifies how an action of X_{\pm} extends to tensor products.

More recently, groupoids and algebroids have been introduced, and these concepts have enlarged the notion of symmetry. Roughly speaking, it is like going from the family of transformations of an infinite tiling to the transformation group of a finite tiling.

References

- [1] Majid, S. 1995. *Foundations of Quantum Group Theory*, Cambridge: Cambridge University Press
- [2] Majid, S. 2000. Quantum groups and concommutative geometry. *Journal of Mathematical Physics*, 41: 3892-942
- [3] Gracia-Bondia, J.M., Lizzi, F., Marmo, G. & Vitale, P. 2002. Infinitely many star products to play with. *JHEP*, 0204:026
- [4] Majid, S. 2002. *A Quantum Groups Primer*. London Mathematical Society Lecture Note Series, 292

3.6 Sklyanin Algebra

One of the most powerful methods for studying the exactly solvable models of quantum and statistical physics is the quantum inverse problem method. The problem of *enumerating* the discrete quantum systems which can be solved by the quantum inverse problem method reduces to the problem of enumerating the operator-valued functions $L(u)$ that satisfy the equation (here $L' \equiv L \otimes 1$ and $L'' \equiv 1 \otimes L$)

$$R(u-v)L'(u)L''(v) = L''(v)L'(u)R(u-v) \quad (3.6.1)$$

for a fixed solution $R(u)$ of the quantum Yang–Baxter equation

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v). \quad (3.6.2)$$

In the classical theory, Eq. (3.6.1) is replaced by (here curly brackets with a comma $\{ , \}$ denote the Poisson bracket, and $[A, B]_{\pm} \equiv AB \pm BA$ for given matrices A and B)

$$\{L'(u), L''(v)\} = [r(u-v), L'(u)L''(v)]_{-}, \quad (3.6.3)$$

while Eq. (3.6.2) becomes the classical Yang–Baxter equation

$$[r_{12}(u-v), r_{13}(u)]_{-} + [r_{12}(u-v), r_{23}(v)]_{-} + [r_{13}(u), r_{23}(v)]_{-} = 0. \quad (3.6.4)$$

Equations (3.6.1) and (3.6.3) provide a systematic procedure for obtaining completely integrable lattice approximations to various continuous completely integrable systems. We here summarize the classical and quantum investigations of Sklyanin, with emphasis on the associated algebras.

(i) *Classical Theory.* Let $r(u)$ be the simplest solution of Eq. (3.6.4):

$$r(u) = \sum_{\alpha=1}^3 w_{\alpha}(u) \sigma_{\alpha} \otimes \sigma_{\alpha}, \quad (3.6.5)$$

where σ_{α} are the Pauli matrices and the coefficients $w_{\alpha}(u)$ can be expressed in terms of the Jacobi elliptic functions as

$$w_1(u) = \rho \frac{1}{\text{sn}(u, k)}, \quad w_2(u) = \rho \frac{\text{dn}(u, k)}{\text{sn}(u, k)}, \quad w_3(u) = \rho \frac{\text{cn}(u, k)}{\text{sn}(u, k)}, \quad (3.6.6)$$

with fixed values of $\rho > 0$ and $k \in [0, 1]$. On looking for solutions of Eq. (3.6.3) in the form

$$L(u) = S_0 + i \sum_{\alpha=1}^3 w_\alpha(u) S_\alpha \sigma_\alpha, \quad (3.6.7)$$

one eventually finds the following quadratic algebra of Poisson brackets for the variables S_α :

$$\{S_\alpha, S_0\} = 2J_{\beta\gamma} S_\beta S_\gamma, \quad (3.6.8)$$

where the right-hand side carries indices β, γ summed over and ranging from 1 to 3, while

$$\{S_\alpha, S_\beta\} = -2S_0 S_\gamma, \quad (3.6.9)$$

where $\alpha \neq \beta \neq \gamma$. The constants $J_{\alpha\beta}$ make it possible to express that the coefficients w_α lie on a quadric, i.e.

$$w_\alpha^2 - w_\beta^2 = J_{\alpha\beta}. \quad (3.6.10)$$

Equations (3.6.8) and (3.6.9) define a quadratic homogeneous Poisson brackets Lie algebra, here denoted by \mathcal{P} . Such equations can be presented in a neater and generalized way upon remarking that, given the differential form (here $i, j = 0, 1, 2, 3$)

$$F = d(a_i x_i^2) \wedge d(b_j x_j^2) = 2(a_i b_j - b_i a_j) x_i x_j dx_i \wedge dx_j, \quad (3.6.11)$$

the contraction of F with a volume element expressed in contravariant form defines a Poisson tensor Λ such that $\Lambda^{lm} = \varepsilon^{lmjk} F_{jk}$. The tensor Λ defines therefore the following Poisson structure on \mathbf{R}^4 :

$$\Lambda = \varepsilon_{ijkl} (a_i b_j - b_i a_j) x_i x_j \partial_k \wedge \partial_l, \quad (3.6.12)$$

and hence we get the Poisson bracket

$$\{x_k, x_l\} = \varepsilon_{kl ij} (a_i b_j - b_i a_j) x_i x_j. \quad (3.6.13)$$

In particular, on setting $b_0 = 0, b_1 = b_2 = b_3 = 1, a_0 = 1$, we get the Sklyanin bracket

$$\{x_k, x_l\} = \varepsilon_{jkl} x_0 x_j, \quad \{x_k, x_0\} = \varepsilon_{jkl} (a_j - a_l) x_j x_l, \quad (3.6.14)$$

where now $j, k, l = 1, 2, 3$.

(ii) *Quantum Theory*. Let $R(u)$ be the solution of Eq. (3.6.2) given by

$$R(u) = 1 + \sum_{\alpha=1}^3 W_{\alpha}(u) \sigma_{\alpha} \otimes \sigma_{\alpha}, \quad (3.6.15)$$

with coefficients $W_{\alpha}(u)$ lying on the algebraic curve (it is a standard notation, in this context, not to write the index γ on the right-hand side)

$$\frac{W_{\alpha}^2 - W_{\beta}^2}{W_{\gamma}^2 - 1} = \mathbf{J}_{\alpha\beta} \quad (3.6.16)$$

and expressed in the form

$$W_1(u) = \frac{\operatorname{sn}(i\eta, k)}{\operatorname{sn}(u + i\eta, k)}, \quad (3.6.17)$$

$$W_2(u) = \frac{\operatorname{dn}}{\operatorname{sn}}(u + i\eta, k) \frac{\operatorname{sn}}{\operatorname{dn}}(i\eta, k), \quad (3.6.18)$$

$$W_3(u) = \frac{\operatorname{cn}}{\operatorname{sn}}(u + i\eta, k) \frac{\operatorname{sn}}{\operatorname{cn}}(i\eta, k). \quad (3.6.19)$$

In analogy with the classical theory, one looks for the solution $\mathbf{L}(u)$ of Eq. (3.6.1) in the form

$$\mathbf{L}(u) = \mathbf{S}_0 + \sum_{\alpha=1}^3 W_{\alpha}(u) \mathbf{S}_{\alpha}, \quad (3.6.20)$$

where the variables \mathbf{S}_{α} are found to obey the commutation relations

$$\left[\mathbf{S}_{\alpha}, \mathbf{S}_0 \right]_{-} = -i \mathbf{J}_{\beta\gamma} \left[\mathbf{S}_{\beta}, \mathbf{S}_{\gamma} \right]_{+}, \quad (3.6.21)$$

$$\left[\mathbf{S}_{\alpha}, \mathbf{S}_{\beta} \right]_{-} = i \left[\mathbf{S}_0, \mathbf{S}_{\gamma} \right]_{+}. \quad (3.6.22)$$

Equations (3.6.21) and (3.6.22) generate a two-sided ideal I in the free associative algebra \mathcal{A} whose four generators are the variables \mathbf{S}_{α} .

If we set $\eta = \rho h$ in Eqs. (3.6.17)–(3.6.19) and pass to the limit of vanishing Planck constant, we obtain the limiting relations

$$W_{\alpha}(u) = i h w_{\alpha}(u) + O(h^2), \quad (3.6.23)$$

$$R(u) = 1 + i h r(u) + O(h^2), \quad (3.6.24)$$

$$\mathbf{J}_{\alpha\beta} = h^2 J_{\alpha\beta} + O(h^4). \quad (3.6.25)$$

On using the expansions (3.6.23)–(3.6.25) and assuming that the quantum quantities \mathbf{S}_α become the corresponding classical quantities S_α when $h \rightarrow 0$, according to the prescription

$$\mathbf{S}_0 \sim hS_0, \quad \mathbf{S}_\alpha \sim S_\alpha,$$

the equations (3.6.1), (3.6.2) and (3.6.21), (3.6.22) for the quantum theory are transformed into the equations (3.6.3), (3.6.4), and (3.6.8), (3.6.9) for the classical theory. Moreover, $\mathbf{L}(u) \sim hL(u)$ and hence the quadratic Poisson bracket algebra \mathcal{P} defined by (3.6.8) and (3.6.9) is the classical limit of the quotient algebra $\mathcal{F} \equiv \frac{\mathcal{A}}{\mathcal{I}}$.

Relevant examples of irreducible, finite-dimensional, self-adjoint representations of the algebra \mathcal{F} are given, in two dimensions, by the Pauli matrices, i.e. $\mathbf{S}_0 = 1$ and $\mathbf{S}_\alpha = \sigma_\alpha$, while in three dimensions, on defining the \mathbf{J}_α through (cf. Eq. (3.6.16))

$$\mathbf{J}_{\alpha\beta} = -\frac{\mathbf{J}_\alpha - \mathbf{J}_\beta}{\mathbf{J}_\gamma}, \quad (3.6.26)$$

one finds

$$\mathbf{S}_0 = \begin{pmatrix} \mathbf{J}_3 & 0 & \mathbf{J}_1 - \mathbf{J}_2 \\ 0 & \mathbf{J}_1 + \mathbf{J}_2 - \mathbf{J}_3 & 0 \\ \mathbf{J}_1 - \mathbf{J}_2 & 0 & \mathbf{J}_3 \end{pmatrix}, \quad (3.6.27)$$

$$\mathbf{S}_1 = \sqrt{2\mathbf{J}_2\mathbf{J}_3} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad (3.6.28)$$

$$\mathbf{S}_2 = \sqrt{2\mathbf{J}_3\mathbf{J}_1} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad (3.6.29)$$

$$\mathbf{S}_3 = 2\sqrt{\mathbf{J}_1\mathbf{J}_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.6.30)$$

The representation expressed by Eqs. (3.6.27)–(3.6.30) is self-adjoint only when $\mathbf{J}_\alpha > 0$.

References

- [1] Sklyanin, E.K. 1982. Some algebraic structures connected with the Yang–Baxter equation. *Functional Analysis and Its Applications*, 16: 263–70

- [2] Quano, Y.H. 1991. Generalized Sklyanin algebra. *Modern Physics Letters*, A6: 3635-40
- [3] Majid, S. 1993. Braided matrix structure of the Sklyanin algebra and of the quantum Lorentz group. *Communications in Mathematical Physics*, 156: 607-38
- [4] Quano, Y.H. 1994. Sklyanin algebra and integrable lattice models. *International Journal of Modern Physics*, A9: 2245-81
- [5] Grabowski, J., Marmo, G. & Perelomov, A.M. 1993. Poisson structures: towards a classification. *Modern Physics Letters*, A8: 1719-33

Acknowledgments. The authors acknowledge partial financial support by PRIN 2002 *SINTESI*.